On one cyclic system and its geometrical interpretation

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Introduction

At mathematical contests the problems in the solution of cyclic equation systems or inequality systems can be found. We call the cyclic set of equations such a set where the equations are produced by cyclic interchange of variables.

We will not delve into the definitions but proceed immediately to the action.

Problem 1 . *Solve the set of equations*

$$\left\{\begin{array}{lcl} x_1^2 - ax_1 + 1 & = & x_2^2 \\ x_2^2 - ax_2 + 1 & = & x_3^2 \\ x_3^2 - ax_3 + 1 & = & x_1^2 \end{array}\right. \tag{1}$$

for all values of $a \neq 0$ parameter.

Constraint $a \neq 0$ is natural because it is evident that with $a = 0$, the system has no solutions.

It can be easily seen that the $x_1 = x_2 = x_3 = \frac{1}{a}$ triple is the system solution. We will call such a solution "trivial". On the other hand, if there exists at least one non-trivial solution of $x_1 = v_0$, $x_2 = u_0$, $x_3 = w_0$, then there are two more solutions obtained by the cyclic interchange of $x_1 = u_0$, $x_2 = w_0$, $x_3 = v_0$ and $x_1 = w_0$, $x_2 = v_0$, $x_3 = u_0$. The search for non-trivial solutions of this set and its generalizations is an interesting research challenge, on whose solution we would like to dwell.

I

Before proceeding further, let us have a look at the following problem.

Problem 2 (M770, [1]) *The base of the triangular pyramid $PABC$ is the equilateral triangle ABC. Prove that the pyramid $PABC$ will be regular if the angles PAB, PBC, PCA are congruent.*

By having made some additional constructions on the plane(!), we can fit the problem solution into a couple of passages.

Solution.(S.A. Valerianov) The base side length can be considered equal to 1. Let $x_1 \geq x_2 \geq x_3 > 0$ be the lengths of PA, PB, PC edges, and $\angle PAB = \angle PBC = \angle PCA = \alpha$. Let us plot α angle on the plane. On one angle arm we will plot the $OK = 1$ intercept, while on the other arm we will plot $OP = x_1$, $OQ = x_2$, $OR = x_3$ intercepts, see Fig. 1. The triangles produced are congruent to the pyramid face so that $RK = x_1$, $PK = x_2$, $QK = x_3$. Out of the RQK triangle we have $x_1 = RK < QR + QK = x_2 - x_3 + x_3 = x_2$, i.e. $x_1 < x_2$ and it is a contradiction. Thus, $x_1 = x_2 = x_3$.

At first glance, this geometrical problem does not have any relation to the set of cyclic equations. However, it is not the case. If the cosine theorem is applied to the PAB, PBC, PCA triangles and if we take $a = 2\cos\alpha$, $0 < |a| < 2$, then we will exactly have the initial set of equations

$$\left\{\begin{array}{lcl} x_1^2 - ax_1 + 1 & = & x_2^2 \\ x_2^2 - ax_2 + 1 & = & x_3^2 \\ x_3^2 - ax_3 + 1 & = & x_1^2. \end{array}\right.$$

Pictorial geometrical interpretation of the set for $0 < |a| < 2$ parameter values shows that when x_1, x_2, x_3 are simultaneous positive integers, only one trivial solution exists. The result can be obtained by "tinkering" with the system.

Solution 2 technique. At first, let is divide all equations by $a^2 \neq 0$ and substitute the $x = \frac{x_1}{a}$, $y = \frac{x_2}{a}$, $z = \frac{x_3}{a}$ variables. We will arrive at the following set

$$\left\{\begin{array}{lcl} x^2 - x + b & = & y^2 \\ y^2 - y + b & = & z^2 \\ z^2 - z + b & = & x^2, \end{array}\right. \tag{2}$$

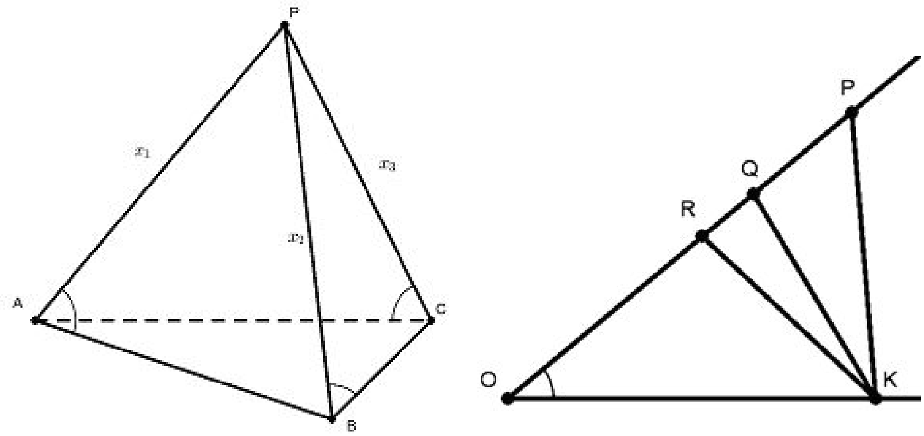

Figure 1: a),b)

where $b = \frac{1}{a^2} > 0$.

Having added all equalities, we directly obtain $x + y + z = 3b$. Let us subtract b^2 out of each equation member and expand into factors

$$\left\{\begin{array}{rcl} x^2 - x + b - b^2 & = & y^2 - b^2 \\ y^2 - y + b - b^2 & = & z^2 - b^2 \\ z^2 - z + b - b^2 & = & x^2 - b^2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} (x-b)(x+b-1) & = & (y-b)(y+b) \\ (y-b)(y+b-1) & = & (z-b)(z+b) \\ (z-b)(z+b-1) & = & (x-b)(x+b). \end{array}\right.$$

Let us multiply equations

$$(x-b)(x+b-1)(y-b)(y+b-1)(z-b)(z+b-1) = (x-b)(x+b)(y-b)(y+b)(z-b)(z+b).$$

Note that if at least one of the variables is equal to b, we will have the trivial solution. Let us prove that alternative triples of integers, that might serve as the system solutions, do not exist.

Indeed, let $x \neq b$, $y \neq b$, $z \neq b$. Let us reduce similar factors in both equation members and obtain

$$(x+b-1)(y+b-1)(z+b-1) = (x+b)(y+b)(z+b).$$

Having opened the brackets, having performed elementary transformations and taking into account that $x + y + z = 3b$, we arrive at the equation $xy + xz + yz = -(3b-1)^2$. It is evident that the equation does not have positive solutions. Thus, if $0 < x$, $0 < y$, $0 < z$, the only existing solution is $x = y = z = b$.

Analysis of the last equation shows that non-trivial solution of the system is also available, and in this case some of the variables must take the negative values.

II

Geometrical interpretation apart, we will attempt to solve the set, using algebraic methods. In this process we will eliminate unnecessary constraints on variables and parameters.

We will focus on the solution of set (2) obtained out of set (1) by substitution of the variables.

As we have already noted, adding of all equalities results in $x+y+z = 3b$, hence $z = 3b-x-y$. Further on,

$$\left\{\begin{array}{rcl} x^2 - x + b & = & y^2 \\ y^2 - y + b & = & (3b-x-y)^2 \\ z & = & 3b - x - y \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x^2 - x + b & = & y^2 \\ y(6b-2x-1) & = & (3b-x)^2 - b \\ z & = & 3b - x - y \end{array}\right. .$$

1) If $6b - 2x - 1 = 0$, then $x = 3b - \frac{1}{2}$, then $b = \frac{1}{4} = x = y = z$.

2) If $6b - 2x - 1 \neq 0$, then $y = \frac{(3b-x)^2-b}{6b-2x-1}$.

We have

$$x^2 - x + b = \left(\frac{(3b-x)^2 - b}{6b - 2x - 1}\right)^2$$

$$(x^2 - x + b)(6b - 2x - 1)^2 - \left((3b - x)^2 - b\right)^2 = 0.$$

We already know that there is a trivial solution for the set, therefore, $x = b$ is the root of the equation obtained. Consequently, the equation left-hand member is divisible by $x - b$. We have

$$(x - b)(3x^3 - 9bx^2 - 3x - 27xb^2 + 18bx + 81b^3 - 54b^2 + 13b - 1) = 0.$$

In order to find other roots, cubic equation

$$x^3 - 3bx^2 + (-9b^2 + 6b - 1)x + 27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3} = 0 \tag{3}$$

should be solved.

Usually, the Cardano formulas are proposed for the solution of cubic equations. Certainly, the students who attend mathematical circles and enrichment classes and, undoubtedly, the mathematical Olympiad participants, can be aware of the Cardano formulas. But the basic difficulty in application of these formulas emerges in the case when the cubic equation has three real roots. In this seemingly not complicated case, some operations with complex numbers should be performed in order to find real solutions, and they are too complicate for the school students. Probably for this reason the Cardano formulas are not included into school curricula.

If we count the discriminant of this cubic equation, we have

$$\triangle = (-3b)^2\left(-9b^2 + 6b - 1\right)^2 - 4(-9b^2 + 6b - 1)^3 - 4(-3b)^3\left(27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3}\right) +$$

$$+18(-3b)(-9b^2 + 6b - 1)\left(27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3}\right) - 27\left(27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3}\right)^2 =$$

$$= (12b^2 - 6b + 1)^2 > 0.$$

So, we find ourselves in the case with three real roots. We will bypass the above described complexities, using complex numbers. To do this, we want to propose the alternative method of solving such cubic equations that is quite readily understandable by school students.

III

Now, we would like to offer a short theoretical digression based on the paper [2].

Let us consider the auxiliary cubic equation

$$Ay^3 - 3By^2 - 3Ay + B = 0, \tag{4}$$

where $A \neq 0$ and A, B are real numbers. Let us define: $0 \leq \phi \leq \pi$, $\sin\phi = \frac{B}{\sqrt{A^2+B^2}}$, $\cos\phi = \frac{A}{\sqrt{A^2+B^2}}$, $\tan\phi = \frac{B}{A}$. Under such conditions the ϕ angle is defined unambiguously. We can show that the following numbers are the roots of equation (4)

$$y_1 = \tan\frac{\phi}{3}, y_2 = \tan\frac{\phi + 2\pi}{3}, y_3 = \tan\frac{\phi + 4\pi}{3}.$$

By checking we use the identity:

$$\tan(3\alpha) = \frac{\tan^3\alpha - 3\tan\alpha}{3\tan^2\alpha - 1}.$$

We insert $y = \tan\frac{\phi+2k\pi}{3}$ into the left part of equation (4), where $k = 0, 1, 2$. We have:

$$A\left(\tan\frac{\phi+2k\pi}{3}\right)^3 - 3B\left(\tan\frac{\phi+2k\pi}{3}\right)^2 - 3A\tan\frac{\phi+2k\pi}{3} + B = 0 \Leftrightarrow$$

$$A\left(\tan^3\frac{\phi+2k\pi}{3} - 3\tan\frac{\phi+2k\pi}{3}\right) = B\left(3\tan^2\frac{\phi+2k\pi}{3} - 1\right) \Leftrightarrow$$

$$\frac{B}{A} = \frac{\tan^3\frac{\phi+2k\pi}{3} - 3\tan\frac{\phi+2k\pi}{3}}{3\tan^2\frac{\phi+2k\pi}{3} - 1} = \tan(\phi+2k\pi) \Leftrightarrow \tan\phi = \frac{B}{A}.$$

QED.

Let us assume that cubic equation

$$x^3 + px^2 + qx + r = 0 \tag{5}$$

written in general perms, has three real roots. Consequently, its discriminant is positive:

$$\triangle = p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2 > 0.$$

Since

$$\triangle = -4\left(q - \frac{p^2}{3}\right)^3 - 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 > 0,$$

then the inequality $q - \frac{p^2}{3} < 0$ is satisfied. Under the indicated conditions, equation (5)can be reduced to equation (4). To do so, we substitute $x = my + n$ into equation (5). We have

$$(my+n)^3 + p(my+n)^2 + q(my+n) + r = 0.$$

To obtain the required equation $Ay^3 - 3By^2 - 3Ay + B = 0$ we will open brackets and equate the coefficients under similar powers of y. Now we have the set of four equations with four unknown values of m, n, A, B that can be easily expressed in terms of p, q, r.

$$\left\{\begin{array}{rcl} m^3 &=& A \\ m^2(3n+p) &=& -3B \\ m(3n^2+2pn+q) &=& -3A \\ n^3+pn^2+qn+r &=& B \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} A &=& m^3 \\ n &=& \frac{9r-pq}{2(p^2-3q)} \\ m &=& \pm\sqrt{-\frac{3n^2+2pn+q}{3}} \\ B &=& n^3+pn^2+qn+r. \end{array}\right.$$

Note that we can choose m using two ways, it can also be expressed in terms of discriminant:

$$m = \pm\sqrt{-\frac{3n^2+2pn+q}{3}} = \pm\frac{\sqrt{3\triangle}}{2(3q-p^2)}.$$

We will resort to one of the two modalities for m and taking $A = m^3$,$B = n^3 + pn^2 + qn + r$, we have the equation of the form (4), where the algorithm for finding the y_1, y_2, y_3 roots is already known to us. Then, basing on the $x = my + n$ formula, we find all solutions of equation (5).

As it can be seen, the proposed method for the solution of cubic equations does not require the application of complex numbers. Instead, the technique makes use of trigonometric functions that are easily understandable by school students.

Now, let us put the presented method into practice and solve the equation (3).

IV

Thus, we look for the solutions of equation

$$x^3 - 3bx^2 + (-9b^2 + 6b - 1)x + 27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3} = 0.$$

For our equation we have

$$p = -3b, q = -9b^2 + 6b - 1, r = 27b^3 - 18b^2 + \frac{13}{3}b - \frac{1}{3}.$$

Earlier, we have already found the discriminant for this cubic equation

$$\triangle = p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2 = (12b^2 - 6b + 1)^2.$$

Now, we only have to insert the indicated expressions into the formulas

$$n = \frac{9r - pq}{2(p^2 - 3q)} = 3b - \frac{1}{2}, m = \frac{\sqrt{3\triangle}}{2(p^2 - 3q)} = \frac{\sqrt{3}}{6}(12b^2 - 6b + 1),$$

$$A = m^3 = \frac{\sqrt{3}}{72}(12b^2 - 6b + 1)^3, B = n^3 + pn^2 + qn + r = -\frac{4b - 1}{24}.$$

So, the roots of equation (3) are the numbers

$$x = m \tan \frac{\phi + 2k\pi}{3} + n = \frac{\sqrt{3}}{6}(12b^2 - 6b + 1) \tan \frac{\phi + 2k\pi}{3} + 3b - \frac{1}{2},$$

where $k = 0, 1, 2$ and

$$\sin\phi = \frac{B}{\sqrt{A^2 + B^2}} = \frac{1 - 4b}{\sqrt{\frac{1}{3}(12b^2 - 6b + 1)^6 + (1 - 4b)^2}}.$$

Taking the trivial solution into account, we see that our system (1) has only four solutions.

Let us illustrate our computations for the value of $b = \frac{1}{2}$ parameter. We solve the system

$$\begin{cases} x^2 - x + \frac{1}{2} &= y^2 \\ y^2 - y + \frac{1}{2} &= z^2 \\ z^2 - z + \frac{1}{2} &= x^2, \end{cases}$$

that gives cubic equation $x^3 - \frac{3}{2}x^2 - \frac{1}{4}x + \frac{17}{24} = 0$. Than

$$\sin\phi = -\frac{\sqrt{3}}{2}, \phi = -\frac{\pi}{3}, x = \frac{\sqrt{3}}{6} \tan \frac{-\frac{\pi}{3} + 2k\pi}{3} + 1, k = 0, 1, 2.$$

Hence, the solutions of system (1) with $a = \sqrt{2}$ are four triples

$$(x_1, x_2, x_3) = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{\sqrt{6}}{6} \tan\frac{\pi}{9} + \sqrt{2}, \frac{\sqrt{6}}{6} \tan\frac{5\pi}{9} + \sqrt{2}, \frac{\sqrt{6}}{6} \tan\frac{11\pi}{9} + \sqrt{2}\right), \right.$$

$$\left(\frac{\sqrt{6}}{6} \tan\frac{5\pi}{9} + \sqrt{2}, \frac{\sqrt{6}}{6} \tan\frac{11\pi}{9} + \sqrt{2}, -\frac{\sqrt{6}}{6} \tan\frac{\pi}{9} + \sqrt{2}\right),$$

$$\left. \left(\frac{\sqrt{6}}{6} \tan\frac{11\pi}{9} + \sqrt{2}, -\frac{\sqrt{6}}{6} \tan\frac{\pi}{9} + \sqrt{2}, \frac{\sqrt{6}}{6} \tan\frac{5\pi}{9} + \sqrt{2}\right) \right\}.$$

V

We would like to develop the success gained and attempt to generalize cyclic set (1). The easiest way to reach the generalization is by increasing the number of equations and variables. The generalization is obtained quite easily, but will we succeed in solving the generalized system? Below we show what can take place in case of four equations and four variables.

Problem 3 . *For all values of $a \neq 0$ parameter solve the set of equations*

$$\left\{\begin{array}{lcl} x_1^2 - ax_1 + 1 & = & x_2^2 \\ x_2^2 - ax_2 + 1 & = & x_3^2 \\ x_3^2 - ax_3 + 1 & = & x_4^2 \\ x_4^2 - ax_4 + 1 & = & x_1^2. \end{array}\right. \tag{6}$$

.

Solution. Let us divide all equations by $a^2 \neq 0$, then assume that $b = \frac{1}{a^2} > 0$, and substituting the variables of $y_i = \frac{x_i}{a}$ we have

$$\left\{\begin{array}{lcl} y_1^2 - y_1 + b & = & y_2^2 \\ y_2^2 - y_2 + b & = & y_3^2 \\ y_3^2 - y_3 + b & = & y_4^2 \\ y_4^2 - y_4 + b & = & y_1^2. \end{array}\right.$$

It follows immediately from the set that $y_1 + y_2 + y_3 + y_4 = 4b$.

Having subtracted the third equation from the first one, and the fourth equation from the second one, we have

$$\left\{\begin{array}{r} (y_1 - y_3)(y_1 + y_3 - 1) = (y_2 - y_4)(y_2 + y_4) \\ (y_2 - y_4)(y_2 + y_4 - 1) = -(y_1 - y_3)(y_1 + y_3). \end{array}\right. \tag{7}$$

Let us multiply the equations, transfer them to the left-hand member and factorize

$$(y_1 - y_3)(y_2 - y_4)\left[(y_1 + y_3 - 1)(y_2 + y_4 - 1) + (y_1 + y_3)(y_2 + y_4)\right] = 0$$

$$(y_1 - y_3)(y_2 - y_4)\left[2(y_1 + y_3)(y_2 + y_4) - (y_1 + y_2 + y_3 + y_4) + 1\right] = 0$$

$$(y_1 - y_3)(y_2 - y_4)\left[2(y_1 + y_3)(y_2 + y_4) - 4b + 1\right] = 0.$$

1) If $y_1 = y_3$, then it follows from (7) that $y_2 = y_4$. And vice versa, if $y_2 = y_4$, then $y_1 = y_3$. Our system is reduced to the set of two equations

$$\left\{\begin{array}{lcl} y_1^2 - y_1 + b & = & y_2^2 \\ y_2^2 - y_2 + b & = & y_1^2. \end{array}\right.$$

We have $y_1 + y_2 = 2b \Leftrightarrow y_2 = 2b - y_1$,

$$y_1^2 - y_1 + b = (2b - y_1)^2 \Leftrightarrow (y_1 - b)(4b - 1) = 0.$$

If $b = \frac{1}{4}$, then we have the family of solutions, $y_2 = \frac{1}{2} - y_1$. Otherwise, $y_1 = y_2 = b$.

2) If $y_1 \neq y_3$, $y_2 \neq y_4$, then $2(y_1 + y_3)(y_2 + y_4) - 4b + 1 = 0$. Denote $u = y_1 + y_3$,$v = y_2 + y_4$, and we have

$$\left\{\begin{array}{lcl} u + v & = & 4b \\ 2uv & = & 4b - 1. \end{array}\right.$$

Whence

$$\left\{\begin{array}{lcl} u & = & \frac{4b - \sqrt{16b^2 - 8b + 2}}{2} \\ v & = & \frac{4b + \sqrt{16b^2 - 8b + 2}}{2} \end{array}\right. \text{ or } \left\{\begin{array}{lcl} u & = & \frac{4b + \sqrt{16b^2 - 8b + 2}}{2} \\ v & = & \frac{4b - \sqrt{16b^2 - 8b + 2}}{2}. \end{array}\right.$$

Further on,

$$\left\{\begin{array}{rcl} y_1^2-y_1+b &=& y_2^2 \\ y_2^2-y_2+b &=& (u-y_1)^2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} y_1^2-y_1+b &=& y_2^2 \\ y_1^2-y_1+b+y_2^2-y_2+b &=& y_2^2+u^2-2uy_1+y_1^2 \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{rcl} y_1^2-y_1+b &=& y_2^2 \\ y_2 &=& 2b-u^2+(2u-1)y_1. \end{array}\right.$$

Thus,

$$y_1^2-y_1+b=(2b-u^2+(2u-1)y_1)^2$$

$$4u(u-1)y_1^2+(2(2u-1)(2b-u^2)+1)y_1+(2b-u^2)^2-b=0.$$

If $u=0$ or $u=1$ we have and obtain the identity of $0\equiv 0$, i.e. y_1 is any value. Otherwise,

$$D=((2(2u-1)(2b-u^2)+1)^2-4\cdot 4u(u-1)((2b-u^2)^2-b)$$

$$y_1=\frac{-((2(2u-1)(2b-u^2)+1)\pm\sqrt{D}}{8u(u-1)}, y_2=2b-u^2+(2u-1)y_1,$$

$$y_3=u-y_1, y_4=4b-u-y_2.$$

Finally we find

$$y_1=\frac{4b-\sqrt{16b^2-8b+2}+\sqrt{2\sqrt{16b^2-8b+2}(4b-1+\sqrt{16b^2-8b+2})}}{4},$$

$$y_2=\frac{4b+\sqrt{16b^2-8b+2}-\sqrt{2\sqrt{16b^2-8b+2}(-4b+1+\sqrt{16b^2-8b+2})}}{4},$$

$$y_3=\frac{4b-\sqrt{16b^2-8b+2}-\sqrt{2\sqrt{16b^2-8b+2}(4b-1+\sqrt{16b^2-8b+2})}}{4},$$

$$y_4=\frac{4b+\sqrt{16b^2-8b+2}+\sqrt{2\sqrt{16b^2-8b+2}(-4b+1+\sqrt{16b^2-8b+2})}}{4}.$$

Other non-trivial solutions we get by the cyclic interchange.

As we can see, using the appropriate pooling of variables, we succeeded to reduce the cyclic set of four equations to the sequential decision of several quadratic equations.

As before, we will explicitly calculate the roots for the value of $b=\frac{1}{2}$ parameter. We have ,

$$y_1=\frac{2-\sqrt{2}+\sqrt{4+2\sqrt{2}}}{4}, y_2=\frac{2+\sqrt{2}-\sqrt{4-2\sqrt{2}}}{4},$$

$$y_3=\frac{2-\sqrt{2}-\sqrt{4+2\sqrt{2}}}{4}, y_4=\frac{2+\sqrt{2}+\sqrt{4-2\sqrt{2}}}{4}.$$

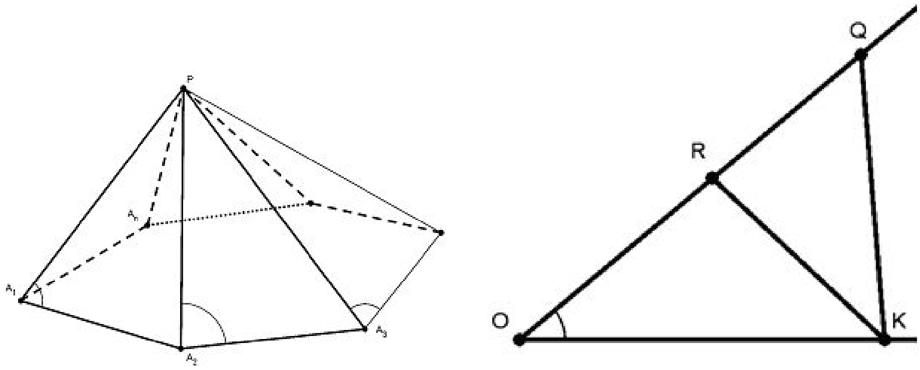

Figure 2: a),b)

VI

We can compose similar geometrical problem for the case of four variables as we did for the case of n variables.

Problem 4 *n-gon $A_1A_2\dots A_n$ with equal lengths of all sides is the base of $PA_1A_2\dots A_n$ pyramid (n-gon is not necessarily regular). Prove that if all PA_iA_{i+1}, PA_nA_1 angles are congruent, for $i = 1, 2, \dots, n-1$, then the $PA_1A_2\dots A_n$ pyramid is regular.*

Solution. Let us assume that the lengths of the base sides are equal to 1, i.e. $A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n = A_nA_1 = 1$.

Let $PA_i = x_i > 0$, $i = 1, 2, \dots, n$ and $\angle PA_1A_2 = \angle PA_1A_2 = \dots = \angle PA_{n-1}A_n = \angle PA_nA_1 = \alpha$. If lateral edges are not equal, then there are edges of minimal and maximal lengths. Without losing generality, we can assume that $PA_1 = x_1$ is the minimal length edge. Let $PA_r = x_r$ be the maximal length edge. Let us consider the corresponding pyramid faces, which have said edges lying opposite α angle, i.e. the PA_nA_1 and $PA_{r-1}A_r$ faces. Now let us proceed similarly to the solution of problem 2.

Let us plot α angle on the plane. Then we plot $OK = 1$ intercept on one arm of the angle, whereas on the other arm we will plot $OQ = x_n$, $OR = x_{r-1}$ intercepts, see Fig.4. The Figure displays the case of $OQ = x_n > OR = x_{r-1}$. (The case of $OQ = x_n < OR = x_{r-1}$ is examined similarly). The triangles produced are congruent to the pyramid faces and so that $QK = x_1$, $RK = x_r$. From the RQK triangle we have $x_r = RK < QR + QK = x_n - x_{r-1} + x_1$, then $x_r + x_{r-1} < x_n + x_1$. Since x_r is the maximal length edge, then $x_r > x_n$, and since x_1 is the minimal length edge, then $x_{r-1} > x_1$, so we have arrived at the contradiction. Thus, $x_1 = x_r$,and as long as the minimal and the maximal edge lengths coincide, all pyramid lateral edges are equal.

It remains to show that the pyramid base is a regular n-gon. To do this, it is sufficient to drop PH, the pyramid height. Since all lateral edges are equal, then all PHA_i, right triangles, are equal. Consequently, all HA_i segments are equal, and H is the circumcenter of $A_1A_2\dots A_n$ n-gon. And since the lengths of all n-gon sides are equal, the figure is regular. QED.

Actually, we have found and proved the novel property of a regular pyramid!!!

VII

We are aware that in case of an arbitrary n there always exists the trivial solution $x_1 = x_2 = \dots = x_n = \frac{1}{a}$ of the system. Besides, if a non-trivial solution also exists, we can find some more solutions obtained using cyclic shift. And though we have not yet solved the cyclic system for an arbitrary n, it only means that our research can be further pursued at all times .

Problem 5 (for research) . *Solve the below set of equations for* $n \geq 5$ *and all values of* $a \neq 0$ *parameter*

$$\begin{cases} x_1^2 - ax_1 + 1 & = & x_2^2 \\ x_2^2 - ax_2 + 1 & = & x_3^2 \\ & \dots & \\ x_n^2 - ax_n + 1 & = & x_1^2. \end{cases}$$

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